PT-Symmetric, Quasi-Exactly Solvable matrix Hamiltonians

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Abstract

Matrix quasi exactly solvable operators are considered and new conditions are determined to test whether a matrix differential operator possesses a or several finite dimensional invariant vector spaces. New examples of 2×2-matrix quasi exactly solvable operators are constructed with the emphasis set on PT-symmetric Hamiltonians.

1 Introduction

Quasi exactly solvable (QES) operators refer to a class of linear operators (typically of Schrödinger type) which preserve a finite dimensional subspace of the Hilbert space on which they act [1, 2, 3]. Is most of the examples known, QES operators can be transformed into operators preserving a space of polynomials of given degree after a suitable change of variable and change of functions (also called "gauge transformations"). The operators preserving a space of polynomials therefore play an important role in the study of QES operators. In the case of scalar equations, the gauge transformation consists in factorizing the ground state out of the wave function, the change of variable can be performed in a straighforward way (at least formally because it leads usually to elliptic integrals).

In the case of coupled equations, where the operators appear in the form of a matrix whose components are differential operators, the construction of the gauge transformation setting the operator in a form which manifestly preserves a vector space where component are polynomials, appear more tricky, see e.g. [4, 5].

In the second section of this paper, we establish a set of algebraic conditions to test wether $n \times n$ matrix-valued operators of a certain type preserve a vector space of n-uple of polynomials with component of definite degrees. We work with 2×2 matrices but the method can be extended to higher dimensions. This new method was tested on all QES known matrix equations. In the other sections, we take advantage of these conditions and construct several new families of QES systems where the emphasis is set on PT-symmetric invariance. This original issue for the mathematical framework of quantum mechanics was proposed in [6] and developped is several subsequent papers but, to our knowledge, it has not been studied in the context of coupled systems of Schrodinger equations.

In Sect. 3 we propose several matrix extentions of the Razhavi operator. Scalar Razhavi-types of potentials were considered recently to produce examples of PT-invariant, non-hermitian potentials with real eigenvalues [7, 8]. Here, we develop matrix extensions of them both with trigonometric and hyperbolic potentials.

In Sect. 4 we obtain a matrix generalisation of the QES example of PT-symmetric hamiltonian with an anharmonic potential of degree four [6] and reconsidered recently [9]. Finally, in Sect. 5, we show how the problem of section 4 can be transformed into a system of recurence equations in the spirit of [10].

2 Matrix QES Operator

In this section, we propose a general test to check whether a 2×2 matrix differential operator H (depending of the variable x and of the derivatives d^n/dx^n), preserves a vector space whose components are polynomials of suitable degrees in x. We will consider a family of operators H which can be decomposed according to

$$H = H_1 + H_0 + \dots, \tag{1}$$

The diagonal components of H_1 are differential operators of degree 1 (i.e acting on a generic polynomial of degree n in x they increase the degree by one unit) and the off-diagonal elements $(H_1)_{12}$ and $(H_1)_{21}$ are respectively proportional to x^δ and $x^{\delta'}$, with $\delta = 0, 1, 2$ and $\delta' \equiv 2 - \delta$. The operators $H_0, \cdots$ have lower degrees in all their components than the corresponding components in H_1. The dots in (1) represent operators with lower degree ordered according to the same rule.

Most of the QES matrix operators known [5, 13, 12] can be reduced to the form above after a suitable gauge transformation and change of variable. The different components of H_1, H_0 contain several constant parameters which label the physical coupling constants in case where H is an Hamiltonian operator.

Now, we try to obtain the conditions of H_1 and H_0 (i.e. conditions on the coupling constants) such that the operator H possesses a finite dimensional invariant subspace of polynomials of the form

$$\mathcal{V} = \text{span}\left\{ \begin{pmatrix} p_n \\ q_m \end{pmatrix} \right\} \quad , \quad n \in \mathbb{N} \quad , \quad m = n - \delta + 1 \tag{2}$$

where p_n, q_m denote polynomials of degree n, m in the variable x. For a generic vector in $\mathcal{V}$ of the form

$$\psi = \begin{pmatrix} \alpha_0 x^n \\ \beta_0 x^{n-\delta+1} \end{pmatrix} + \begin{pmatrix} \alpha_1 x^{n-1} \\ \beta_1 x^{n-\delta} \end{pmatrix} + \ldots , \tag{3}$$

where α_i, β_i are complex parameters, the vector $H\psi$ can be decomposed according to

$$\begin{aligned} H\psi \; &= \text{diag}(x^{n+1}, x^{n-\delta+2}) M_1 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \\ &+ \left(\text{diag}(x^n, x^{n-\delta+1}) \tilde{M}_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + \text{diag}(x^n, x^{n-\delta+1}) M_0 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \right) \\ &+ \text{terms of lower degrees} \quad , \end{aligned} \tag{4}$$

where the constant 2×2 matrices $M_1, \tilde{M}_1$ and M_0 can be obtained after a simple algebra.

The necessary conditions for $\mathcal{V}$ to contain an invariant vector space of the operator H read

$$(i) \qquad M_1 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5}$$

$$(ii) \qquad \tilde{M}_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + M_0 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \propto \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \tag{6}$$

where the second condition has to be fullfilled irrespectively of the values α_1, β_1.

The condition (i) implies $\det M_1 = 0$ and the vector $(\alpha_0, \beta_0)^t$ to be a zero-eigenvalue eigenvector of M_1. This fixes the relative coefficient of the terms of highest degree in $\mathcal{V}$ (see Eq.(2)). The condition (ii) can be fullfilled only if the following conditions hold

$$(ii') \quad M_0 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \Lambda \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \quad , \quad \tilde{M}_1^t \begin{pmatrix} -\beta_0 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \tag{7}$$

where M^t means the transpose matrix of M.

The conditions (i),(ii') allow to reconstruct in a systematic way the invariant vector spaces of all QES operators presented e.g. in [12, 5, 13]; in particular the conditions on the different parameters and the relevant changes of variable now emerge in terms of elementary algebra on matrices.

In order to illustrate this method, we reconstruct the invariant vector space of the QES Hamiltonian [12, 13]

$$H(y) = -\frac{d^2}{dy^2}\mathbb{I}_2 + M_6(y), \tag{8}$$

where $M_6(y)$ is a 2×2 hermitian matrix of the form

$$M_6(y) = \{4p_2^2y^6+8p_1p_2y^4+(4p_1^2-8mp_2+2(1-2\epsilon)p_2)y^2\}\mathbb{I}_2+(8p_2y^2+4p_1)\sigma_3-8mp_2\kappa_0\sigma_1 \tag{9}$$

It is known that after the usual "gauge transformation" of $H(y)$ with a factor

$$\phi(y) = y^\epsilon \exp -\{\frac{p_2}{2}y^4 + p_1y^2\} \quad , \quad \epsilon = 0, 1 \tag{10}$$

and the change of variable $x = y^2$, the new operator $\tilde{H}(x)$ is obtained

$$\tilde{H}(x) = \phi^{-1}(y)H(y)\phi(y)|_{y=\sqrt{x}} \tag{11}$$

For simplicity we assume $\epsilon = 0, p_1 = 0$ in the following. The operator obtained $\tilde{H}(x)$ then reads

$$\tilde{H}(x) = (-4x\frac{d^2}{dx^2} - 2\frac{d}{dx})\mathbb{I}_2 + 8p_2\begin{pmatrix} J_+(m-2) & 0 \\ 0 & J_+(m)\end{pmatrix} - 8mp_2\kappa_0\sigma_1 \tag{12}$$

with $J_+(m) \equiv x^2d_x - mx$. It can be decomposed along the lines of Eq.(1) :

$$\tilde{H}(x) = H_1 + H_0 + H_{-1}, \tag{13}$$

with

$$\begin{aligned} H_1 &= 8p_2\begin{pmatrix} J_+(m-2) & -m\kappa_0 \\ 0 & J_+(m)\end{pmatrix}, \\ H_0 &= 0, \\ H_{-1} &= (-4x\frac{d^2}{dx^2} - 2\frac{d}{dx})\mathbb{I}_2 - 8mp_2\kappa_0\sigma_- \end{aligned} \tag{14}$$

In this case, $(H_1)_{12}$ is a constant(i.e. $\delta = 0$), while $(H_1)_{21} = 0$, in addition the operatot H_0 is zero. The invariant vector space to be looked for is of the form

$$\psi = \begin{pmatrix} \alpha_0x^{m-1} + \alpha_1x^{m-2} + \dots \\ \beta_0x^m + \beta_1x^{m-1} + \dots \end{pmatrix} \tag{15}$$

The determinant of the matrix M_1 is trivially zero and the condition (i) implies $\frac{\alpha_0}{\beta_0} = m\kappa_0$. The first conditions (ii') is trivial since $M_0 = 0$ (as a consequence of $H_0 = 0$). Finally, the second condition (ii') can be easily checked :

$$\begin{aligned} \tilde{M}_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} &= -8p_2\beta_1 \begin{pmatrix} m\kappa_0 \\ 1 \end{pmatrix} = -8p_2\beta_1 \begin{pmatrix} \frac{\alpha_0}{\beta_0} \\ 1 \end{pmatrix} \\ -8p_2\beta_1\beta_0 \begin{pmatrix} m\kappa_0 \\ 1 \end{pmatrix} &= -\beta_1 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \end{aligned} \tag{16}$$

In the following section, we will present several examples of QES matrix operator based on extensions of the scalar Razavi potential.

3 PT invariant non-hermitian matrix Hamiltonian

In [7, 8] PT-invariant models based on the scalar Razhavi potential are analyzed with the emphasis set on the reality properties of the spectrum. This can be done partly in an analytical way because the potentials considered are QES. The authors considerd both, hyperbolic and trigonometric cases, invoking an anti-isospectral transformation [11] to relate the spectra of both types. Here we will consider matrix extensions of these equations and see that several form of the non diagonal elements H_{12} and H_{21} can lead to QES operators. We will first consider periodic potentials, formulated in terms of trigonometric functions. Then an example involving elliptic functions for the potentials will be presented.

3.1 Trigonometric case

From the unidimensional potential studied in [7], we will build a family PT invariant matrix Hamiltonian and use the technique developped the previous section to check its quasi exactly solvability. We start from a general Hamiltonian of the form

$$H = \begin{pmatrix} -\frac{d^2}{dx^2} + (\rho\cos 2x - iM)^2 + A & H_{12} \\ H_{21} & -\frac{d^2}{dx^2} + (\rho\cos 2x - i\tilde{M})^2 + \tilde{A} \end{pmatrix}, \tag{17}$$

where ρ is a free real parameter and $A, \tilde{A}, M, \tilde{M}$ are constant to be specified. There are several forms of H_{12}, H_{21} which lead to QES operators. One can assume $\tilde{M} > M$ without loosing generality. The general properties of the diagonal component of H and of trigonometric fonctions will reveal that QES operators can be constructed by choosing H_{12} according to one of the following form

$$H_{12} = C\cos 2x + D \ \text{ or } \ H_{12} = C\cos x \ \text{ or } \ H_{12} = C\sin x \ \text{ or } \ H_{12} = C\cos x\sin x \tag{18}$$

and similar forms respectively for H_{21} with, however, a priori independent coupling constants for C and D.

In order to reveal the algebraic properties of this family of operators, it is convenient to perform a first gauge transformation according to

$$\begin{aligned}
\tilde{H} &= e^{-\theta\cos 2x}\begin{pmatrix} z^{-\epsilon}(1-z)^{-\phi} & 0 \\ 0 & z^{-\tilde{\epsilon}}(1-z)^{-\tilde{\phi}} \end{pmatrix} H e^{\theta\cos 2x}\begin{pmatrix} z^{\epsilon}(1-z)^{\phi} & 0 \\ 0 & z^{\tilde{\epsilon}}(1-z)^{\tilde{\phi}} \end{pmatrix}, \\
&= \begin{pmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{pmatrix},
\end{aligned} \tag{19}$$

where $z = (\cos 2x + 1)/2$. Further choosing the parameter θ, according to $\theta = i\frac{\rho}{2}$ the components of $\tilde{H}$ are obtained after an algebra:

$$\begin{aligned}
\tilde{H}_{11} &= -4z(1-z)\frac{d^2}{dz^2} + 2(2z-1-4(1-z)\epsilon + 4\phi z)\frac{d}{dz} + \rho^2 - M^2 + 8\phi\epsilon + 2\epsilon + 2\phi + A \\
&\quad -8i\rho(z(1-z)\frac{d}{dz} + \epsilon(1-z) - \phi z + \frac{M-1}{4}(2z-1)) \\
\tilde{H}_{12} &= z^{\tilde{\epsilon}-\epsilon}(1-z)^{\tilde{\phi}-\phi}H_{12}, \\
\tilde{H}_{21} &= z^{\epsilon-\tilde{\epsilon}}(1-z)^{\phi-\tilde{\phi}}H_{21}, \\
\tilde{H}_{22} &= -4z(1-z)\frac{d^2}{dz^2} + 2(2z-1-4(1-z)\tilde{\epsilon} + 4\tilde{\phi} z)\frac{d}{dz} + \rho^2 - M^2 + 8\tilde{\phi}\tilde{\epsilon} + 2\tilde{\epsilon} + 2\tilde{\phi} + \tilde{A} \\
&\quad -8i\rho(z(1-z)\frac{d}{dz} + \tilde{\epsilon}(1-z) - \phi z + \frac{\tilde{M}-1}{4}(2z-1))
\end{aligned} \tag{20}$$

and where we have neglected the singular terms of the form

$$\frac{1-z}{z}2\epsilon(2\epsilon-1) + \frac{z}{1-z}2\phi(2\phi-1) \tag{21}$$

in H_{11} (and a similar terms with $\epsilon \to \tilde{\epsilon}$, $\phi \to \tilde{\phi}$ in H_{22}) since we assume from now on

$$\epsilon(2\epsilon-1) = \phi(2\phi-1) = \tilde{\epsilon}(2\tilde{\epsilon}-1) = \tilde{\phi}(2\tilde{\phi}-1) = 0 \tag{22}$$

The different choices for H_{12} proposed in Eq.(18) now appear to be natural since they will automatically lead to a polynomial expressions in z when the choice of the parameters $\epsilon, \tilde{\epsilon}, \phi, \tilde{\phi}$ is done according to Eq. (22). In the following, we will analyze in details the case $H_{12} = C\sin x\cos x$. The algebraisation corresponding to the three other cases can be done similarly.

In this case, the possible values for the parameters $\epsilon, \tilde{\epsilon}, \phi, \tilde{\phi}$ allow for four algebraisation, for the wave function $\psi = (\psi_1, \psi_2)$, namely :

$$\begin{aligned}
\text{type } i \ &: \quad \psi = (p_n, \sin x\cos x q_{n-1}) \\
\text{type } ii \ &: \quad \psi = (p_{n-1}\sin x\cos x, q_n) \\
\text{type } iii \ &: \quad \psi = (p_n\sin x, q_n\cos x) \\
\text{type } iv \ &: \quad \psi = (p_n\cos x, q_n\sin x)
\end{aligned}$$

where p_n, q_n, etc. denote polynomials of degree n in the variable z.

Acting on an eigenfunction of type (i), the conditions for algebraic solutions are $\epsilon = \phi = 0$, $\tilde{\epsilon} = \tilde{\phi} = 1/2$. The operator $\tilde{H}$ can then be decomposed according to the prescription of Sect. 2, leading to :

$$\tilde{H} = \tilde{H}_1 + \tilde{H}_0 + \tilde{H}_{-1}, \tag{23}$$

whith

$$\tilde{H}_1 = \begin{pmatrix} 8i\rho(z^2\frac{d}{dz} - (\frac{M-1}{2})z) & -Cz^2 \\ \tilde{C} & 8i\rho(z^2\frac{d}{dz} - (\frac{\tilde{M}-3}{2})z \end{pmatrix}, \tag{24}$$

$$\tilde{H}_0 = (4z^2\frac{d^2}{dz^2} + (4-8i\rho)z\frac{d}{dz} + \rho^2)\mathbb{I}_2 + \begin{pmatrix} A' & Cz \\ 0 & 8z\frac{d}{dz} + \tilde{A}' \end{pmatrix} \tag{25}$$

and

$$\tilde{H}_{-1} = \begin{pmatrix} -4z\frac{d^2}{dz^2} - 2\frac{d}{dz} & 0 \\ 0 & -4z\frac{d^2}{dz^2} - 6\frac{d}{dz} \end{pmatrix}. \tag{26}$$

with

$$A' = A - M^2 + 2i\rho(M-1)\ ,\ \tilde{A}' = \tilde{A} + 4 - \tilde{M}^2 + 2i\rho(\tilde{M}-3) \tag{27}$$

Using the parametrisation corresponding to type (i) for the wave function, we can easily obtain the form of the matrices $M_1, \tilde{M}_1, M_0$ and the conditions on the parameters leading to QES operators. In the present case, we got

$$M + \tilde{M} = 4n \quad , \quad (1-4n^2) + M\tilde{M} = \frac{C\tilde{C}}{16\rho^2} \tag{28}$$

The condition involving M_0 fixes the difference between the constant $A, \tilde{A}$, namely

$$A - \tilde{A} = M^2 - \tilde{M}^2 \tag{29}$$

Considering the parametrisation corresponding to type (ii) for the wave function, one can easily obtain $\epsilon = \phi = 1/2$, $\tilde{\epsilon} = \tilde{\phi} = 0$ and the associated operator $\tilde{H}$. The action of $\tilde{H}$ on an eigenfunction of the type (ii) gives after an algebra the matrices $M_1, \tilde{M}_1, M_0$ which lead to the same QES conditions found in the previous case as given by the Eq.(28) and the Eq.(30).

This time, the wave functions of the type (iii) and of the type (iv) correspond respectively to $\epsilon = \tilde{\phi} = 0$, $\tilde{\epsilon} = \phi = 1/2$ and $\epsilon = \tilde{\phi} = 1/2$, $\tilde{\epsilon} = \phi = 0$. After some algebra, one can find the corresponding operators $\tilde{H}$ and also the matrices $M_1, \tilde{M}_1, M_0$ are deduced. For these two types (iii) and (iv), we got the three same QES conditions

$$M + \tilde{M} = 4n + 2 \quad , \quad M\tilde{M} - 4n(n+1) = \frac{C\tilde{C}}{16\rho^2} \tag{30}$$

The condition involving M_0 fixes the difference between the constant $A, \tilde{A}$, namely

$$A - \tilde{A} = M^2 - \tilde{M}^2. \tag{31}$$

It is found that this QES condition is the same for all four types of the wave function.

A consequence of these results is that the equations (17) admits a double algebraisation. Solutions of the types *i* and *ii* exist if the condition (28) is fulfilled and solution of the types *iii* and *iv* if the condition (30) holds.

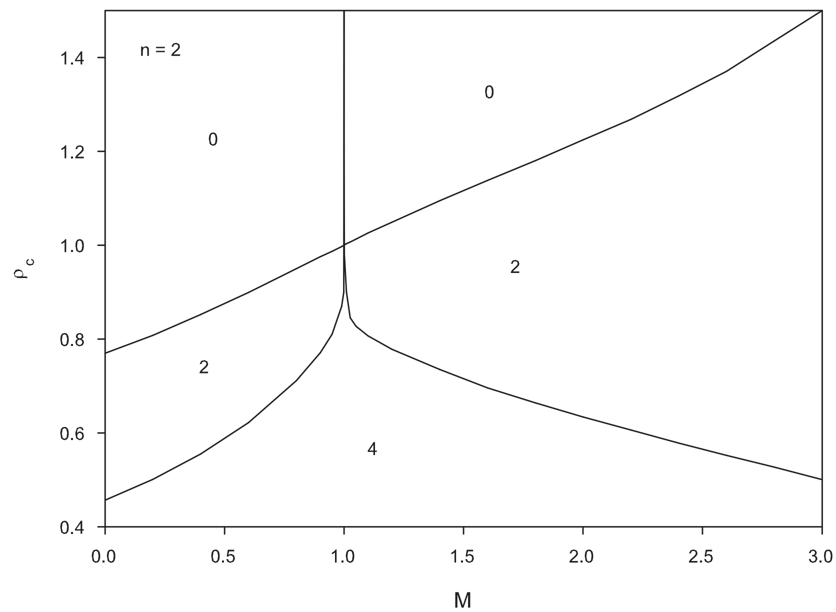

Figure 1: The critical value of ρ as a function of the coupling constant M for the type i solution and $n = 2$. The integers label the number of real algebraic eigenvalues

3.2 Some properties of the spectrum

We studied the algebraic eigenvalues of the equation (17) for the solution of the type i and for $n = 1, 2$. The invariant vector space has dimension $2n$ keeping into account that the condition (5) imposes a constraint on the polynomials. In the case $n = 1$ the two algebraic eigenvalues have the form

$$E = \rho^2 + 2 \pm \sqrt{1 - \rho^2(1 + M)^2} \ . \tag{32}$$

Showing that the algebraic eigenvalues are real only for $|\rho| < 1/|1 + M|$. Similar features are observed namely in [7],[8]. Namely the eigenvalues come out to be real or in complex degenerate pairs. For $n = 2$ the polynomial giving the four algebraic eigenvalues for E is real but rather involved and the solution cannot be expressed in a closed form for generic values of M, ρ. In the case $M = 1$ and $M = 3$, however, we could find explicit solutions :

$$M = 1 \ \ , \ \ E = 4 + \rho^2 \ \ (2 \text{ times}) \ \ , \ \ E = 8 + \rho^2 \pm 8\sqrt{1 - \rho^2}$$

$$M = 3 \ \ , \ \ E = 10 + \rho^2 \pm \sqrt{9 - 4\rho^2} \ \ , \ \ E = 2 + \rho^2 \pm 2\sqrt{1 - 4\rho^2}$$

The plane M, ρ is partitionned into regions admitting 4,2 or 0 algebraic eigenvalues. The critical values of ρ_c are presented as functions of the parameters M for $0 \le M \le 3$. We note in particular that one of the critical line become infinite in the limit $M \to 1$, indicating that in this case there are two real eigenvalues for $\rho > 1$.

3.3 Hyperbolic case

The construction of previous section can also be realized for the case where the trigonometric functions entering in the potentials are replaced by their elliptic counterpart. The discussion of the different algebraisations turn out to be the same. Here however, we will study in detail the algebraic properties of the operator given by

$$H = \begin{pmatrix} -\frac{d^2}{dx^2} - (\rho\cosh 2x - iM)^2 & C(\cosh 2x - 1) + \tilde{C} \\ D(\cosh 2x - 1) + \tilde{D} & -\frac{d^2}{dx^2} - (\rho\cosh 2x - i\tilde{M})^2 \end{pmatrix}, \tag{33}$$

where ρ is a free real parameter. One can assume $\tilde{M} > M$ without loosing generality. The gauge transformation is performed as follows

$$\begin{aligned} \tilde{H} &= \exp(-\theta\cosh 2x) H \exp(\theta\cosh 2x), \\ &= \begin{pmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{pmatrix}, \end{aligned} \tag{34}$$

On further substituting $z = \cosh 2x - 1$ and fixing the constant θ by means of $\theta = \frac{i\rho}{2}$, the different components of $\tilde{H}$ read

$$\begin{aligned} \tilde{H}_{11} &= -4z(z+2)\frac{d^2}{dz^2} - 4(z+1)\frac{d}{dz} - 8i\rho z\frac{d}{dz} - \rho^2 - 4i\rho(z^2\frac{d}{dz} - \frac{M-1}{2}z) \\ &\quad +2i\rho(M-1) + M^2, \\ \tilde{H}_{12} &= Cz + \tilde{C}, \\ \tilde{H}_{21} &= Dz + \tilde{D}, \\ \tilde{H}_{22} &= -4z(z+2)\frac{d^2}{dz^2} - 4(z+1)\frac{d}{dz} - 8i\rho z\frac{d}{dz} - \rho^2 - 4i\rho(z^2\frac{d}{dz} - \frac{\tilde{M}-1}{2}z) \\ &\quad +2i\rho(\tilde{M}-1) + \tilde{M}^2. \end{aligned} \tag{35}$$

Decomposing now the operator $\tilde{H}$ according to Eq.(1), we obtain

$$\tilde{H}_1 = \begin{pmatrix} -4i\rho(z^2\frac{d}{dz} - Nz) & Cz \\ Dz & -4i\rho(z^2\frac{d}{dz} - \tilde{N}z) \end{pmatrix}, \tag{36}$$

where we posed $N = \frac{M-1}{2}$, $\tilde{N} = \frac{\tilde{M}-1}{2}$. The form of $\tilde{H}_0$ and $\tilde{H}_{-1}$ can be obtained easily. Note that $(\tilde{H}_1)_{12} = Cz$ and $(\tilde{H}_1)_{21} = Dz$, so that $\delta = \delta' = 1$ in this case. Refering to the above general case and with

$$\psi = \begin{pmatrix} \alpha_0 z^n \\ \beta_0 z^n \end{pmatrix} + \begin{pmatrix} \alpha_1 z^{n-1} \\ \beta_1 z^{n-1} \end{pmatrix} + \ldots, \tag{37}$$

we can write the vector $\tilde{H}\psi$ according to

$$\begin{aligned} \tilde{H}\psi &= \mathrm{diag}(z^{n+1}, z^{n+1}) M_1 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \\ &\quad + \left(\mathrm{diag}(z^n, z^n)\tilde{M}_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + \mathrm{diag}(z^n, z^n) M_0 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}\right) \\ &\quad + \ldots, \end{aligned} \tag{38}$$

where

$$\begin{aligned}
M_1 &= \begin{pmatrix} -4i\rho(n-N) & C \\ D & -4i\rho(n-\tilde{N}) \end{pmatrix}, \\
\tilde{M}_1 &= \begin{pmatrix} -4i\rho(n-1-N) & C \\ D & -4i\rho(n-1-\tilde{N}) \end{pmatrix}, \\
M_0 &= -(4n^2+8i\rho n+\rho^2)\mathbb{1} + \begin{pmatrix} 4i\rho N+(2N+1)^2 & \tilde{C} \\ \tilde{D} & 4i\rho\tilde{N}+(2\tilde{N}+1)^2 \end{pmatrix}. \quad (39)
\end{aligned}$$

The three necessary conditions for the operator $\tilde{H}$ to have a finite dimensional invariant vector space can then be obtained in a straightforward way, the final results read :

$$N+\tilde{N} = 2n-1 \quad , \quad 16\rho^2(n-N)(n-\tilde{N}) + CD = 0 \quad , \quad \frac{\beta_0}{\alpha_0} = \frac{4i\rho(n-N)}{C} \qquad (40)$$

the equation involving the metric M_0 imposes in turn

$$\tilde{C}\beta_0^2 + 4(N-\tilde{N})(2n+i\rho)\beta_0\alpha_0 - \tilde{D}\alpha_0^2 = 0 \qquad (41)$$

As a result, assuming a choice of the integer n, we end up with a family of QES operators labelled by the parameters N, ρ, C/D and $\tilde{C}$.

Different choices of the non-diagonal interactions H_{12} and H_{21} can be performed which lead to similar conditions between the cosmological constants. We will discuss these possibilities in the framework of periodic potentials (formulated in terms of trigonometric functions). largely discussed in the next section.

4 PT-symmetric QES equation with polynomial potential

In this section, refering to unidimensional operator studied in [9] we will construct a PT-symmetric QES matrix Hamiltonian of the form

$$H = -\frac{d^2}{dx^2}\mathbb{1}_2 + M_4(x) \qquad (42)$$

where $M_4(x)$ is 2×2 PT-symmetric matrix. The above Hamiltonian can be written in terms of components and we choose the potentials of the form :

$$\begin{aligned}
H_{11} &= -\frac{d^2}{dx^2} - x^4 + iAx^3 + Bx^2 + iCx + D, \\
H_{12} &= \omega, \\
H_{21} &= \tilde{\omega}, \\
H_{22} &= -\frac{d^2}{dx^2} - x^4 + i\tilde{A}x^3 + \tilde{B}x^2 + i\tilde{C}x + \tilde{D} \qquad (43)
\end{aligned}$$

In order to reveal the QES property, it is convenient to perform a gauge transformation according to

$$\tilde{H} = \exp(-\alpha x^3 - \beta x^2 - \gamma x) H \exp(\alpha x^3 + \beta x^2 + \gamma x), \tag{44}$$

The gauged Hamiltonian then simplifies considerably if

$$\alpha = -\frac{i}{3}\ ,\ \beta = -\frac{A}{4}\ ,\ \gamma = \frac{i}{2}(B - \frac{A^2}{4})\ ,\ A = \tilde{A}\ ,\ B = \tilde{B}. \tag{45}$$

leading to the following expression :

$$\begin{aligned} \tilde{H} &= -\frac{d^2}{dx^2} - 4\beta x\frac{d}{dx} - 2\gamma\frac{d}{dx} - 6\alpha[(x^2\frac{d}{dx} - mx) + \theta x\sigma_3] \\ &+(-2\beta - \gamma^2) + \mathrm{diag}(D, \tilde{D}) + \omega\sigma_+ + \tilde{\omega}\sigma_-, \end{aligned} \tag{46}$$

where the constants $C, \tilde{C}$ have been redefined according to $C = i(6\alpha(m-\theta) + 6\alpha + 4\beta\gamma)$, $\tilde{C} = i(6\alpha(m+\theta) + 6\alpha + 4\beta\gamma)$.

However, in this form, the occurence of an invariant finite dimensional vector space of function is not yet manifest in the sense that the operator $\tilde{H}$ doesn't preserve the vector space $(P_{m-\theta}, P_{m+\theta})^t$. In order to reveal such a possibility we can apply the technique of the first section. Here we will follow [5] and perform a supplementary transformation on the operator $\tilde{H}$ with the matrix $S = \begin{pmatrix} 1 & \lambda\frac{\partial}{\partial x} \\ 0 & 1 \end{pmatrix}$. After an algebra, we obtain finally the form

$$\begin{aligned} \hat{H} &= S^{-1}\tilde{H}S, \\ &= [-\frac{d^2}{dx^2} + Ax\frac{d}{dx} - i(B - \frac{A^2}{4})\frac{d}{dx} + D + \frac{1}{4}(B - \frac{A^2}{4})^2] - \tilde{\omega}\lambda\frac{d}{dx}\sigma_3 \\ &+2i\mathrm{diag}(J_+(n-2), J_+(n)) + \mathrm{diag}(\frac{A}{2}, -\frac{A}{2}) + \tilde{\omega}\sigma_- - \tilde{\omega}\lambda^2\frac{d^2}{dx^2}\sigma_+ \end{aligned} \tag{47}$$

with $J_+(n) \equiv x^2\frac{d}{dx} - nx$. Here we have set $m = n-1, \tilde{D} = -A + D, \theta = 1$ and fixed the arbitrary parameter λ entering in the gauge transformation by means of $\omega = -2i\lambda n$.

The Hamiltonian $\hat{H}$ manifestly preserves the finite dimensional space $(P_{n-2}, P_n)^t$. Note that $A, B, D, \tilde{\omega}$ are free real parameters, n is a non-negative integer and λ is a free complex parameter.

5 Recurence relations

In this section we will express the formulation of the QES solution in terms of recurence relations to the case of PT-symmetic matrix Hamiltonian. We will see that the eigenvalue equation $H\psi = E\psi$ leads to a system of four terms recurence relations. The solutions ψ are of the form

$$\psi(x) = \exp(-\frac{ix^3}{3} - \frac{Ax^2}{4} + \frac{i}{2}(B - \frac{A^2}{4})x)\begin{pmatrix} \sum_{k=0}^{\infty} P_k(E)x^k \\ \sum_{l=0}^{\infty} Q_l(E)x^l \end{pmatrix} \tag{48}$$

To solve the equation $H\psi = E\psi$ is equivalent to solve the following equation

$$\hat{H} \begin{pmatrix} \sum_{k=0}^{\infty} P_k(E) x^k \\ \sum_{l=0}^{\infty} Q_l(E) x^l \end{pmatrix} = E \begin{pmatrix} \sum_{k=0}^{\infty} P_k(E) x^k \\ \sum_{l=0}^{\infty} Q_l(E) x^l \end{pmatrix} \tag{49}$$

Then the equation above can be transformed into a *fourth*-order recurence relation. It reads

$$A_k \begin{pmatrix} P_k \\ Q_{k+2} \end{pmatrix} + B_k \begin{pmatrix} P_{k-1} \\ Q_{k+1} \end{pmatrix} + C_k \begin{pmatrix} P_{k-2} \\ Q_k \end{pmatrix} + D_k \begin{pmatrix} P_{k-3} \\ Q_{k-1} \end{pmatrix} = 0 \tag{50}$$

where

$$\begin{aligned}
A_k &= \begin{pmatrix} k(k-1) & 0 \\ -\tilde{\omega} & (k+2)(k+1) \end{pmatrix}, \\
B_k &= \begin{pmatrix} [i(B - \frac{A^2}{4}) + \lambda\tilde{\omega}](k-1) & 0 \\ 0 & [-\lambda\tilde{\omega} + i(B - \frac{A^2}{4})](k+1) \end{pmatrix}, \\
C_k &= \begin{pmatrix} -D - \frac{1}{4}(B - \frac{A^2}{4})^2 - A(k-2) - \frac{A}{2} + E & \tilde{\omega}\lambda^2 k(k-1) \\ 0 & -D - \frac{1}{4}(B - \frac{A^2}{4})^2 - Ak + \frac{A}{2} + E \end{pmatrix} \\
D_k &= -2i \begin{pmatrix} (k-n-1) & 0 \\ 0 & (k-n-1) \end{pmatrix},
\end{aligned} \tag{51}$$

In the present case, the recurence relations are of fourth order, contrasting with other cases studied in the litterature [10, 13] where they are of third order. Setting $\omega = \tilde{\omega} = 0$ the two recurence relations decouple and the corresponding equations (e.g. the one for P_k) correspond to the scalar PT-invariant and QES quartic oscillator. It is also of fourth order; as a consequence, both P_0 and P_1 are arbitrary (P_0 fixes the normalisation) and the other $P_k, k \geq 2$ are determined recursively. The construction of the QES eigenvalues associated to this system is not as transparant in in the case of third-order recurence where a common factor, say P_n factorize out of the P_k's, $k > n$. In the present case, the QES eigenvalues are obtained by solving the system

$$P_n(E, P_1) = 0 \quad , \quad P_{n-1}(E, P_1) = 0 \tag{52}$$

which is linear in P_1. These conditions indeed lead to a truncation of the series for $\psi_1(x)$ defined in (48). Coming back to the full system (i.e. with $\omega \neq 0, \tilde{\omega} \neq 0$), it is easy to see that Q_0, Q_1, Q_2, Q_3 remain arbitrary (Q_0 set the normalisation). The QES eigenvalues can be obtained by solving the system

$$\begin{aligned}
P_n(E, Q_1, Q_2, Q_3) = 0 \quad &, \quad P_{n-1}(E, Q_1, Q_2, Q_3) = 0 \\
Q_{n+2}(E, Q_1, Q_2, Q_3) = 0 \quad &, \quad Q_{n+1}(E, Q_1, Q_2, Q_3) = 0
\end{aligned} \tag{53}$$

which turns out to be linear in Q_1, Q_2, Q_3.

6 Conclusions

In this paper, we have proposed a set of simple nesessary and sufficient conditions for matrix-valued operators of a certain type to preserve a vector space of polynomials of fixed degrees. We have seen that the scalar Razhavi potential admits QES matrix extensions of several types. We also constructed a QES, matrix-valued PT invariant Hamiltonian with polynomial potentials. Finally, by taking this last problem as example, we have shown that the coupled differential equations can be transformed into a system coupled recurence equations of fourth order.

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